Structure in Quaternions Corresponding to the 4-Dimensional Tetrahedron

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Abstract

Identify the vertices of the 4-dimensional tetrahedron, or simplex, with unit quaternions. In this paper we show that, under stereographic projection, the nontrivial quotients of these quaternions form a 3-dimensional dodecahedron. Then, we explore the algebraic structure of the aforementioned dodecahedron and investigate the properties and patterns of longer products of the five original unit quaternions and their inverses.

1 Introduction

Let (t, x, y, z) be a point in $\mathbb{R}^4$. Then its corresponding quaternion (represented by a matrix) is

$$\begin{pmatrix} t + ix & y + iz \\ -y + iz & t - ix \end{pmatrix}.$$

It can also be represented as $t + x\hat{i} + y\hat{j} + z\hat{k}$. If t=0, then (t, x, y, z) corresponds to a *pure quaternion.* If $t^2 + x^2 + y^2 + z^2 = 1$, then (t, x, y, z) corresponds to a *unit quaternion.* The product of two quaternions can be obtained by multiplying their matrix representations. In addition, the product of two unit quaternions is also a unit quaternion.

If we let u be a unit quaternion, and take $t = \cos\theta$ and $\sqrt{x^2 + y^2 + z^2} = \sin\theta$, then $u = \cos\theta + \sin\theta u_0$, where $u_0 = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$. Letting q be a pure quaternion corresponding to a point p in $\mathbb{R}^3$, then the map $q \longmapsto uqu^*$ is a rotation of p in $\mathbb{R}^3$ by an angle of 2θ about the axis of Ou_0. Note that the product of two unit quaternions corresponds to the compostion of their respective rotations of $\mathbb{R}^3$.

We will work on the unit sphere $S^3 \in \mathbb{R}^4$ and illustrate the results with the stereographic projection of S^3 onto $\mathbb{R}^3 \cup \{\infty\}$.

2 Points of the 4D tetrahedron

We used the following set of vertices for the 4D tetrahedron:

$$p_1 = (1, 0, 0, 0)$$

$$p_2 = (-\frac{1}{4}, \frac{\sqrt{5}}{4}, \frac{\sqrt{5}}{4}, \frac{\sqrt{5}}{4})$$

$$p_3 = (-\frac{1}{4}, \frac{\sqrt{5}}{4}, -\frac{\sqrt{5}}{4}, -\frac{\sqrt{5}}{4})$$

$$p_4 = (-\frac{1}{4}, -\frac{\sqrt{5}}{4}, -\frac{\sqrt{5}}{4}, \frac{\sqrt{5}}{4})$$

$$p_5 = (-\frac{1}{4}, -\frac{\sqrt{5}}{4}, \frac{\sqrt{5}}{4}, -\frac{\sqrt{5}}{4})$$

3 Obtaining the dodecahedron

The vertices of the 4D tetrahedron are on the unit sphere, so we can identify them with unit quaternions. Let $\mathbf{Q}_0 = \{a, b, c, d, e\}$ be the quaternions obtained from the points p_1, p_2, p_3, p_4 and p_5, respectively. The quaternions corresponding to the vertices of the 3D dodecahedron can be obtained by the following method, which was devised by Ocneanu. We multiply each of the five quaternions by a^{-1} on the left. This will give us the following five quaternions:

$a^{-1}a = 1, a^{-1}b, a^{-1}c, a^{-1}d$, and $a^{-1}e$. Then we repeat this procedure with b^{-1}, c^{-1}, d^{-1}, and e^{-1}, yielding a total of 25 unit quaternions. However, we will disregard all trivial products which give us the identity. This leaves us with 20 distinct unit quaternions (denoted as set $\mathbf{Q}_1$), which we can identify with 20 points on S^3. When appropriate, we will refer to 4D points as quaternions and vice versa. Using the fomula $(x_0, x_1, x_2, x_3) \longmapsto \frac{1}{1+x_0}(x_1, x_2, x_3)$, we stereographically project the points onto $\mathbb{R}^3$. Let $\widetilde{\mathbf{Q}}_1$ represent the set of 20 3D points obtained in this way.

4 Two properties of the sets $\mathbf{Q}_1$ and $\widetilde{\mathbf{Q}}_1$

Now we will show that each of the 20 points on S^3 (namely, points of the form $w_1^{-1}w_2$, where $w_1 \neq w_2 \in \mathbf{Q}_0$) is the same distance from the identity in $\mathbb{R}^4$. Since $\mathbf{Q}_0$ are the vertices of the 4D regular tetrahedron, the distance in $\mathbb{R}^4$ between any pair of vertices is equivalent. From this, we know that the distance is given by

$$d_{\mathbb{R}^4}(w_2^{-1}w_1, I) = d_{\mathbb{R}^4}(w_1, w_2) = C,$$

where C is some constant. Thus, all elements of $\mathbf{Q}_1$ are the same distance from the identity in $\mathbb{R}^4$. Moreover, this implies that they are on the same level curve in $\mathbb{R}^4$, so their stereographic projections in $\mathbb{R}^3$ will have equivalent norms.

Consider two sets of three distinct points, (x_1, x_2, x_3) and (y_1, y_2, y_3), where all $x_i, y_i \in \mathbf{Q}_0$. There exist unique unit quaternions u and v such that $y_i = ux_iv$. Thus,

$$\begin{aligned} &y_1^{-1}y_2 \\ &= (v^{-1}x_1^{-1}u^{-1})(ux_2v) \\ &= v^{-1}x_1^{-1}x_2v \end{aligned}$$

and similarly, $y_3^{-1}y_1 = v^{-1}x_3^{-1}x_1v$. Then

$$\begin{aligned} &d_{\mathbb{R}^4}(y_1^{-1}y_2, y_3^{-1}y_1) \\ &= d_{\mathbb{R}^4}(v^{-1}(x_1^{-1}x_2)v, v^{-1}(x_3^{-1}x_1)v) \\ &= d_{\mathbb{R}^4}(x_1^{-1}x_2, x_3^{-1}x_1) \end{aligned}$$

By the above result, all lines in $\mathbb{R}^4$ with vertices in the form of $(x_1^{-1}x_2, x_3^{-1}x_1)$, where $x_1 \neq x_2 \neq x_3 \in \mathbf{Q}_0$, have equivalent lengths. Moreover, all the vertices of lines with this form are elements of $\mathbf{Q}_1$. Thus, stereographic projection will scale the line lengths by the same factor, resulting in equivalent lines lengths in $\mathbb{R}^3$ also.

5 Proof that $\widetilde{\mathbf{Q}}_1$ gives the vertices of the 3D regular dodecahedron

It is a geometric fact that the 3D regular dodecahedron can be constructed as follows: take a regular tetrahedron inscribed into a sphere and find its negative.

Combined, the two sets of points give vertices which form a regular cube. For each edge of the cube, find a point on the sphere that is equidistant from the two vertices of the edge. Since a cube has 12 edges, there will be 12 such points. Together with the eight vertices of the cube, these 20 points is a set of vertices for the regular 3D dodecahedron. The following proof, which was outlined by Ocneanu, shows that the elements of $\widetilde{\mathbf{Q}}_1$ are precisely these vertices.

Now consider the set $\widetilde{\mathbf{Q}}_1$ as defined above. Each point in $\widetilde{\mathbf{Q}}_1$ corresponds to a product of unit quaternions in $\mathbf{Q}_1$ (i.e. $a^{-1}b, e^{-1}c$). Since a is the identity, the products $a^{-1}b, a^{-1}c, a^{-1}d$, and $a^{-1}e$ are just the original quaternions b, c, d, and e. After stereographic projection, these points will form a 3D regular tetrahedron. We will now refer to points in $\widetilde{\mathbf{Q}}_1$ in terms of their corresponding quaternions in $\mathbf{Q}_1$. Taking the inverse of each vertex, we will obtain another tetrahedron with vertices $b^{-1}a, c^{-1}a, d^{-1}a$, and $e^{-1}a$. Note that these products belong to $\mathbf{Q}_1$. These two tetrahedrons form a cube.

The edges of the cube are of the form $(a^{-1}y_1, y_2^{-1}a)$, where each of the two points is a vertex of the edge and $y_1 \neq y_2 \in \{b, c, d, e\}$. Now we want to find a point that is equidistant from both vertices. By the second property of the previous section, the point $y_1^{-1}y_2$ has this property. Since there are 12 edges, there are 12 such points, which is all the possible combinations of two elements from $\{b, c, d, e\}$. Combined with the eight vertices of cube, these are the 20 vertices of a 3D regular dodecahedron. Moreover, these 20 points are exactly the elements of $\widetilde{\mathbf{Q}}_1$, which completes our proof.

A natural question arises here. What if we had taken the products in the beginning with the inverses on the right side instead of the left? In this case, we would still obtain a dodecahedron. However, we would obtain a set of vertices different from $\widetilde{\mathbf{Q}}_1$.

6 Classifying pairs of points in $\mathbf{Q}_1$ by edge distance of corresponding points in $\widetilde{\mathbf{Q}}_1$

Let $d_{ED}(x_1^{-1}x_2, x_3^{-1}x_4)$ with $x_i \in \mathbf{Q}_0$ represent the minimum number of edges of the 3D dodecahedron which are needed to travel from $x_1^{-1}x_2$ to $x_3^{-1}x_4$. Note that for a point p, $d_{ED}(p, p^{-1}) = 5$, the maximum edge distance between two points in $\widetilde{\mathbf{Q}}_1$. Given two points p_1 and p_2, there exists a unique five edge-length path between p_1 and p_1^{-1} that passes through p_2. Thus, $d_{ED}(p_1^{-1}, p_2) = 5 - d_{ED}(p_1, p_2)$.

6.1 Case 1: $|\{x_1, x_2\} \cap \{x_3, x_4\}| = 2$

Suppose $x_1 = x_3$ and $x_2 = x_4$. Then $d_{ED}(x_1^{-1}x_2, x_3^{-1}x_4) = 0$ trivially. It follows that $d_{ED}(x_1^{-1}x_2, x_4^{-1}x_3) = 5$.

6.2 Case 2: $|\{x_1, x_2\} \cap \{x_3, x_4\}| = 1$

Suppose $x_1 = x_3$. Then by an isometry theorem for $\mathbb{R}^4$, $d_{\mathbb{R}^4}(x_1^{-1}x_2, x_3^{-1}x_4) = d_{\mathbb{R}^4}(x_2, x_4)$. Recall that x_2 and x_4 are vertices of the 4D tetrahedron, and after stereographic projection, they become vertices of the 3D tetrahedron, which are the diagonal points on each face of the cubes inherent in the dodecahedron. These diagonal points have an edge distance of 3, and since $x_1^{-1}x_2, x_3^{-1}x_4, x_2, x_4 \in \mathbf{Q}_1$, the stereographic projection will scale the distance between the first two points and the distance between the latter two points with the same factor. Thus, $d_{ED}(x_1^{-1}x_2, x_3^{-1}x_4) = d_{ED}(x_2, x_4) = 3$. Similarily, if $x_2 = x_4$, $d_{ED}(x_1^{-1}x_2, x_3^{-1}x_4) = d_{ED}(x_1^{-1}, x_3^{-1}) = 3$. Now let $x_1 = x_4$ or $x_2 = x_3$. These are the complementary cases to the above two situations. In both cases, $d_{ED}(x_1^{-1}x_2, x_3^{-1}x_4) = 5 - 3 = 2$.

6.3 Case 3: $|\{x_1, x_2\} \cap \{x_3, x_4\}| = 0$

Now let all four x_i's be distinct. Only one pair of complementary edge distances remains: one and four. In $\mathbb{R}^4$, four points of a positive (negative) orientation can only be rotated to four points which also have positive (negative) orientation. We can calculate the orientation of the points (x_1, x_2, x_3, x_4) by taking the determinant of the 4×4 matrix, $(x_1|x_2|x_3|x_4)$. A positive (negative) determinant corresponds to positive (negative) orientation. For example, $\det(a|b|c|d)$ is negative, so (a, b, c, d) has negative orientation. By straightforward calculations, we find that $d_{\mathbb{R}^3}(a^{-1}b, c^{-1}d) < d_{\mathbb{R}^3}(a^{-1}b, d^{-1}c)$. Thus, $d_{ED}(a^{-1}b, c^{-1}d) = 1$, and accordingly, $d_{ED}(a^{-1}b, d^{-1}c) = 4$. Furthermore, if (x_1, x_2, x_3, x_4) has negative orientation, then $d_{ED}(x_1^{-1}x_2, x_3^{-1}x_4) = 1$. If (x_1, x_2, x_3, x_4) has positive orientation, then $d_{ED}(x_1^{-1}x_2, x_3^{-1}x_4) = 4$.

6.4 Algebraic structure of dodecahedron edges

One property of special interest is that if we take the points *abcd* and attach the remaining element of $\mathbf{Q}_0$ (namely, e) on the end of the list, then the list becomes an even permutation of *abcde*. Recall that (a, b, c, d) has negative orientation. Then if $x_1x_2x_3x_4x_5$ is an even permutation of *abcde*, (x_1, x_2, x_3, x_4) has negative orientation also, so $d_{ED}(x_1^{-1}x_2, x_3^{-1}x_4) = 1$. In other words, the edges of the dodecahedron have vertices $(x_1^{-1}x_2, x_3^{-1}x_4)$, where all x_i's are distinct elements from $\mathbf{Q}_0$ and $x_1x_2x_3x_4x_5$ (where x_5 is the remaining element of $\mathbf{Q}_0$) is an even permutation of *abcde*.

There are 120 permutations of *abcde*, but we want only the 60 even permutations. But since $(x_1^{-1}x_2, x_3^{-1}x_4) = (x_3^{-1}x_4, x_1^{-1}x_2)$, we are left with 30 permutations, which precisely gives us the vertices of the 30 edges of the dodecahedron.

7 Algebraic structure of dodecahedron faces

It turns out that there is another useful property of even permutations of *abcde*. Let $ax_1x_2x_3x_4$ be an even permutation of *abcde* with a fixed. Then $(a^{-1}x_1,\ x_2^{-1}x_3,\ x_4^{-1}a,\ x_1^{-1}x_2,\ x_3^{-1}x_4)$ are the vertices (in clockwise order) of one of the 12 pentagonal faces of the dodecahedron. For example, *abcde* represents the face with vertices $\{a^{-1}b,\ c^{-1}d,\ e^{-1}a,\ b^{-1}c,\ d^{-1}e\}$. Notice that the vertices are made up of the cycle *abcde* repeated twice. The vertices of the other 11 faces are also just even permutations of *abcde* (with a fixed) repeated twice.

Once again, we start with 120 permuations of *abcde*. There are 60 even permutations, and when we fix a at the beginning, we are left with the 12 distinct permutations that give us the faces of the dodecahedron.

8 Rotational symmetries of the dodecahedron

The dodecahedron has 60 (counterclockwise) rotational symmetries. Besides the identity rotation, there are four rotations about the centers of each of 6 pairs of opposite faces (multiples of $\frac{2\pi}{5}$), two rotations about each of the 10 pairs of opposite vertices (multiples of $\frac{2\pi}{3}$), and one π rotation about the center of each of the 15 pairs of opposite edges. Each of the 60 rotations is a bijective mapping of $\mathbf{Q}_0$ to $\mathbf{Q}_0$.

8.1 Observations on rotations about faces

Let a face be determined by $x_0x_1x_2x_3x_4$, an even permutation of *abcde* with x_0 fixed as a. Then a $\frac{2\pi}{5}$ (counterclockwise) rotation of the dodecahedron about the center of the face can be described by the map $x_i \longmapsto x_{(i+3)mod 5}$. Since every vertex of the dodecahedron is of the form $w^{-1}y$, where $w \neq y \in \{x_0, x_1, x_2, x_3, x_4\}$, the map tells us precisely which vertex rotates into which other vertex. If we examine the face *aecbd*, a $\frac{2\pi}{5}$ rotation can be described by the permutation *bdaec*. More specifically, it stipulates that the five vertices of the face determined by *aecbd* rotates according to the following map

$$(a^{-1}e,\ c^{-1}b,\ d^{-1}a,\ e^{-1}c,\ b^{-1}d)$$

$$\downarrow$$

$$(b^{-1}d,\ a^{-1}e,\ c^{-1}b,\ d^{-1}a,\ e^{-1}c)$$

Rotations of $\frac{4\pi}{5}$, $\frac{6\pi}{5}$, $\frac{8\pi}{5}$ can be obtained by composing rotations of $\frac{2\pi}{5}$.

8.2 Observations on rotations about edges

Given an edge with vertices $(x_1^{-1}x_2, x_3^{-1}x_4)$, where $x_1, x_2, x_3, x_4 \in \mathbf{Q}_0$, a π rotation of the dodecahedron about the center of that edge can be described by the map $\{x_1 \longmapsto x_3, x_2 \longmapsto x_4, x_3 \longmapsto x_1, x_4 \longmapsto x_2, x_5 \longmapsto x_5\}$, where x_5 is the remaining element of $\mathbf{Q}_0$. Since every vertex of the dodecahedron is of the form $w^{-1}y$, where $w \neq y \in \{x_1, x_2, x_3, x_4, x_5\}$, the map tells us precisely which vertex rotates into which other vertex. For example, if we want to rotate by π about the edge $(a^{-1}b, c^{-1}d)$, the vertices $b^{-1}e$ and $d^{-1}a$ would be rotated into $d^{-1}e$ and $b^{-1}c$, respectively.

8.3 Observations on rotations about vertices

Given a vertex $x_1^{-1}x_2$, a $\frac{2\pi}{3}$ (counterclockwise) rotation of the dodecahedron about the vertex can be defined by the following method: We first choose $z_1 \in \mathbf{Q}_0$, $z_1 \neq x_1, x_2$. Then we choose $z_2 \in \mathbf{Q}_0$, $z_2 \neq x_1, x_2, z_1$ so that $(x_1^{-1}x_2, z_1^{-1}z_2)$ are the vertices of an edge of the dodecahedron. Then the rotation can be described by the map $\{x_1 \longmapsto x_1, x_2 \longmapsto x_2, z_1 \longmapsto z_3, z_2 \longmapsto z_1, z_3 \longmapsto z_2\}$, where z_3 is the remaining element of $\mathbf{Q}_0$. Since every vertex of the dodecahedron is of the form $w^{-1}y$, where $w \neq y \in \{x_1, x_2, z_1, z_4, z_3\}$, the map tells us precisely which vertex rotates into which other vertex. For example, if we want to rotate by $\frac{2\pi}{3}$ about the vertex $b^{-1}d$, we can choose to use the edge with vertices $(b^{-1}d, a^{-1}e)$. In this case, $z_3 = c$. Then the map $\{b \longmapsto b, d \longmapsto d, a \longmapsto c, e \longmapsto a, c \longmapsto e\}$ will rotate the vertices $b^{-1}e$ and $d^{-1}a$ into $b^{-1}a$ and $d^{-1}c$, respectively. Rotations of $\frac{4\pi}{3}$ can be obtained by composing two rotations of $\frac{2\pi}{3}$.

9 Longer quaternion products

Beyond the dodecahedron, we also examined larger products of the elements in $\mathbf{Q}_0$ and their inverses. As stated before, $\mathbf{Q}_1$ represents nontrivial products of the form $x^{-1}y$ where $x, y \in \mathbf{Q}_0$. We define $\mathbf{Q}_i$ to be the set of quaternions which are the result of the noncollapsing products (i.e. no pairwise cancellations) of i elements of $\mathbf{Q}_1$. For example,

$$\mathbf{Q}_2 = \{xy \mid x, y \in \mathbf{Q}_1\} = \{w^{-1}xy^{-1}z \mid w, x, y, z \in \mathbf{Q}_0\}$$

$$\mathbf{Q}_3 = \{xyz \mid x, y, z \in \mathbf{Q}_1\} = \{u^{-1}vw^{-1}xy^{-1}z \mid u, v, w, x, y, z \in \mathbf{Q}_0\}.$$

9.1 $\mathbf{Q}_2$

When we examined $\widetilde{\mathbf{Q}}_2$, the stereographic projection of $\mathbf{Q}_2$, we noticed that the points can be divided into 5 layers by their norms. Those norms, beginning with the smallest, are

$$\left\{ \sqrt{\frac{5}{41}\left(19 - 8\sqrt{5}\right)}, \sqrt{\frac{5}{11}}, \sqrt{\frac{5}{3}}, \sqrt{\frac{5}{41}\left(19 + 8\sqrt{5}\right)}, \sqrt{15} \right\}. \tag{1}$$

Since the points that make up each of the layers of $\widetilde{\mathbf{Q}}_2$ lie on a sphere in $\mathbb{R}^3$ centered at the origin, the pre-image of those points under stereographic projection lie on a level curve of $S^3 \subset \mathbb{R}^4$. This is the same as saying that points of $\widetilde{\mathbf{Q}}_2$ with the same norm correspond to $\mathbf{Q}_2$ points with the same t coordinate.

To discuss the different layers of $\widetilde{\mathbf{Q}}_2$, we'll use $\widetilde{\mathbf{Q}}_2[i]$ to refer to the ith norm layer of $\widetilde{\mathbf{Q}}_2$, as ordered in (**??**). For example, $\widetilde{\mathbf{Q}}_2[2] = \left\{ x \in \widetilde{\mathbf{Q}}_2 \mid ||x|| = \sqrt{\frac{5}{11}} \right\}$.

9.2 Observations about the norm layers of $\widetilde{\mathbf{Q}}_2$

At least two of the levels in $\widetilde{\mathbf{Q}}_2$, namely, $\widetilde{\mathbf{Q}}_2[2]$ and $\widetilde{\mathbf{Q}}_2[3]$, can be found on $\widetilde{\mathbf{Q}}_1$, the dodecahedron. Let L be the set of all lines on the dodecahedron which connect points on the same face, but which are not edges of the dodecahedron. Then $\widetilde{\mathbf{Q}}_2[2]$, after proper normalization, is exactly the intersection points of these lines on the faces of the dodecahedron. Thus, $\widetilde{\mathbf{Q}}_2[2]$, is the set of points which form small pentagons on each of the faces of the dodecahedron. If these lines are parameterized by $t \in [0, 1]$, these points occur at exactly the point $t = \frac{2}{1+\sqrt{5}}$, the inverse of the golden ratio.

It turns out, on the other hand, that $\widetilde{\mathbf{Q}}_2[3]$ can be obtained, after proper normalization, by intersecting the lines in L with circles centered on the faces of the dodecahedron so that the points obtained actually form regular decagons on the faces of the dodecahedron. We checked that these points gave us decagons by computing the dot products and distances between the points to show that they formed a regular polygon.

9.3 Algebraic characterization of the layers of $\widetilde{\mathbf{Q}}_2$

Each of the six $\widetilde{\mathbf{Q}}_2[i]$'s has a distinct alegraic form, which is related to the characterization of the six distinct edge distances in the dodecahedron, $\widetilde{\mathbf{Q}}_1$.

Each of the points in $\mathbf{Q}_2$ are of the form $u^{-1}vx^{-1}y$ where $u, v, x, y \in \mathbf{Q}_0$. Each of the $\widetilde{\mathbf{Q}}_2[i]$'s lies on a sphere of some radius in $\mathbb{R}^3$, and thus, the pre-image of these points under stereographic projection corresponds to the slice of $S^3 \subset \mathbb{R}^4$ with some constant t coordinate. Since we are dealing only with unit quaternions, we know that all the elements of $\mathbf{Q}_2$ are on the unit sphere in $\mathbb{R}^4$. If we can then determine the distance between $(1, 0, 0, 0) \in \mathbb{R}^4$, the identity quaternion, and the elements of $\mathbf{Q}_2$, we will essentially know the t coordinate for each of the quaternions in $\mathbf{Q}_2$. This is because the intersection of the unit sphere centered at the origin with another sphere of some radius centered at $(1, 0, 0, 0)$ is exactly the slice of $S^3 \subset \mathbb{R}^4$ with some constant t coordinate. With knowledge of this t value, we will be able to identify which of the $\widetilde{\mathbf{Q}}_2[i]$'s the quaternions belong to.

Now let's try to characterize the t coordinate of the elements of $\mathbf{Q}_2$. If $a \in \mathbf{Q}_2$, then it is of the form $u^{-1}vx^{-1}y$ with $u, v, x, y \in \mathbf{Q}_0$. We know that the distance from a to the origin is 1 because u, v, x, y are unit quaternions. We'll use the notation $d_{\mathbb{R}^4}(a, (0, 0, 0, 0)) = ||a|| = 1$. Now consider the distance from a to the identity. We multiply on the left with u and on the right with y^{-1}, and by the quaternion isometry theorem, we have that

$$d_{\mathbb{R}^4}(a, I) = d_{\mathbb{R}^4}(u^{-1}vx^{-1}y, I) = d_{\mathbb{R}^4}(vx^{-1}, uy^{-1}). \tag{2}$$

The right hand side of this equation seems to be very close to what we were considering when we were looking at the edge distances of the dodecahedron, except that the inverse elements are in the wrong positions. If we do a little more multiplication we get

$$d_{\mathbb{R}^4}(a, I) = d_{\mathbb{R}^4}(vx^{-1}, uy^{-1}) = d_{\mathbb{R}^4}(u^{-1}v, y^{-1}x). \tag{3}$$

Now the right hand side of this equation is exactly the distance between the two points of the dodecahedron, $u^{-1}v$ and $y^{-1}x$. In this way, we have defined a map from elements of $\mathbf{Q}_2$ to pairs of vertices making up edges in the dodecahedron.

$$u^{-1}vx^{-1}y \mapsto (u^{-1}v, y^{-1}x) \quad u, v, x, y \in \mathbf{Q}_0 \tag{4}$$

All elements of $\mathbf{Q}_2$ which map to vertices of the same edge distance have the same distance to the identity in $\mathbb{R}^4$, and therefore lie on the same norm layer of $\widetilde{\mathbf{Q}}_2$.

Using this map and the previous characterization of the edge distances on the dodecahedron, we can give the following characterization of the $\widetilde{\mathbf{Q}}_2[i]$'s:

- $\widetilde{\mathbf{Q}}_2[1]$ has elements of the form $u^{-1}vx^{-1}y$ where $u, v, x, y \in \mathbf{Q}_0$ are all different and $(u, v, x, y, *)$ is an odd permutation of (a, b, c, d, e).
- $\widetilde{\mathbf{Q}}_2[2]$ has elements of the form $u^{-1}xu^{-1}y$ where $u, x, y \in \mathbf{Q}_0$ are all different.
- $\widetilde{\mathbf{Q}}_2[3]$ has elements of the form $u^{-1}xy^{-1}u$ where $u, x, y \in \mathbf{Q}_0$ are all different.
- $\widetilde{\mathbf{Q}}_2[4]$ has elements of the form $u^{-1}vx^{-1}y$ where $u, v, x, y \in \mathbf{Q}_0$ are all different and $(u, v, x, y, *)$ is an even permutation of (a, b, c, d, e).
- $\widetilde{\mathbf{Q}}_2[5]$ has elements of the form $x^{-1}yx^{-1}y$ where $x, y \in \mathbf{Q}_0$ are different.

Note that the elements of $\widetilde{\mathbf{Q}}_2[1]$ map to exactly the pairs of vertices on the dodecahedron with an edge distance of one. The elements of $\widetilde{\mathbf{Q}}_2[2]$ correspond to the pairs of vertices of the dodecahedron with an edge distance of two. This pattern continues for the other $\widetilde{\mathbf{Q}}_2[i]$'s. This is a result of the way we chose the stereographic projection. Using the map $(x_0, x_1, x_2, x_3) \longmapsto \frac{1}{1+x_0}(x_1, x_2, x_3)$ we are projecting from the "south pole" of the sphere S^3. If we had projected from the north pole, the order of the edge distances that each $\widetilde{\mathbf{Q}}_2[i]$ corresponded to would be inverted.

References

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